

Logarithmic corrections to three-dimensional black holes and de Sitter spaces

Y.S. Myung*

Relativity Research Center and School of Computer Aided Science, Inje University, Gimhae 621-749, Korea

Abstract

We calculate logarithmic corrections to the Bekenstein-Hawking entropy for three-dimensional BTZ black hole with $J = 0$ and Kerr-de Sitter (KdS) space with $J = 0$ including the Schwarzschild-de Sitter (SdS) solution due to thermal fluctuations. It is found that there is no distinction between the event horizon of the BTZ black hole and the cosmological horizon of KdS space. We obtain the same correction to the Cardy formula for BTZ, KdS, and SdS cases. We discuss AdS/CFT and dS/ECFT correspondences in connection with logarithmic corrections.

*E-mail address: ysmyoung@physics.inje.ac.kr

I. INTRODUCTION

Recently there are several works which show that for a large class of AdS black holes, the Bekenstein-Hawking entropy receives logarithmic corrections due to thermodynamic fluctuations [1–6]. The corrected formula takes the form

$$S = S_0 - \frac{1}{2} \ln(C_v T^2) + \dots, \quad (1)$$

where C_v is the specific heat of a given gravitational system at constant volume and S_0 denotes the Bekenstein-Hawking entropy. Here an important point is that for Eq.(1) to make sense, C_v should be positive. However, the d-dimensional (dD) Schwarzschild black hole which is asymptotically flat has a negative specific heat of $C_v^{Sch} = -(d-2)S_0$ and thus its canonical ensemble is unstable [7,8]. This means that the Schwarzschild black hole is never in thermal equilibrium with its environment. But introducing a negative cosmological constant $\Lambda = -(d-1)(d-2)/2\ell^2$ could make the specific heat positive and render Eq. (1) applicable. The 5D Schwarzschild-AdS black hole with the event horizon $r_{EH}^2 = \ell^2(-1 + \sqrt{1 + 4m/\ell^2})/2$ belongs to this category [9,10]. In the limit of large $\Lambda(\ell \rightarrow 0, m \gg \ell^2)$, one has a large AdS black hole with $C_v^{SAdS} \simeq 3S_0$, while in the limit of small $\Lambda(\ell \rightarrow \infty, m \ll \ell^2)$, one finds a small AdS black hole which leads to the Schwarzschild black hole with $C_v^{SAdS} \simeq -3S_0$. This implies that for a black hole with a large mass, the presence of a large negative cosmological constant is sufficient to make the specific heat positive.

In other words, a large black hole in asymptotically AdS space has a positive specific heat, while a small black hole in asymptotically AdS space has a negative specific heat. Roughly speaking, one considers the AdS space as a confining box. Then a large black hole in a small box (large AdS black hole) can be in thermal equilibrium and gives a positive specific heat as a whole. On the contrary, if the box is large and unbounded as in the small AdS black hole (\simeq Schwarzschild black hole), its specific heat becomes negative and thus this system could not be in thermal equilibrium. Similarly one considers a cosmological horizon in 5D Schwarzschild-de Sitter space [11]. In the limit of large $\Lambda(\ell \rightarrow 0)$ one finds a positive specific heat of the pure de Sitter space $C_v^{SdS} \simeq 3S_0$, while in the limit of small $\Lambda(\ell \rightarrow \infty)$ one recovers a negative specific heat of the Schwarzschild black hole as $C_v^{SdS} \simeq -3S_0$. The cosmological horizon of 5D hyperbolic topological de Sitter (HTdS) space [12] has the two limits [13,14]. Also we expect a similar property in topological Reissner-Nordstrom de Sitter space [15].

However, the above two limits do not exist for a lower-dimensional gravity system. In other words, this system is simple in compared with 5D gravity systems. A class of 3D gravitational systems have no a singularity but a conical defect (excess) inside the single horizon. Also A(dS)/(E)CFT correspondences can be easily confirmed in connection with logarithmic corrections. In this work, we study 3D AdS black hole and de Sitter spaces which give positive specific heats and thus logarithmic corrections to the entropy could be achieved. These are the BTZ black hole with $J = 0$ and Kerr-de Sitter (KdS) space with $J = 0$ including the Schwarzschild-de Sitter (SdS) solution. All of them have single horizons.

II. 3D ADS BLACK HOLE

It is believed that a higher-dimensional black hole in asymptotically flat spacetime should have a spherical horizon. When introducing a negative cosmological constant, a higher-dimensional black hole could have a non-spherical horizon. We call this the topological black hole [16,17]. However, in three dimensions we could not construct various horizon geometries because one-dimensional horizon is locally flat and is equivalent to each other. Instead, we introduce an AdS black hole, BTZ black hole with $J = 0$ [18] as

$$ds_{AdS}^2 = -h_{AdS}(r)dt^2 + \frac{1}{h_{AdS}(r)}dr^2 + r^2d\theta^2, \quad (2)$$

where a metric function $h_{AdS}(r)$ is given by

$$h_{AdS}(r) = -\mu_{AdS} + \frac{r^2}{\ell^2} \quad (3)$$

with a reduced mass $\mu_{AdS} = 8G_3\mathcal{M}_{AdS}$ for a massive point particle located at $r = 0$ inside the event horizon. Here we choose a circle (S^1) for the horizon geometry. The event horizon is given by

$$r_{EH} = \ell\sqrt{\mu_{AdS}}. \quad (4)$$

Relevant thermodynamic quantities: free energy (F), Bekenstein-Hawking entropy (S_0), Hawking temperature (T_H), specific heat ($C_v = (dE/dT)_V$) and energy (ADM mass: $E = M$) are given by

$$\begin{aligned} F &= -\frac{\mu_{AdS}}{8G_3}, \quad S_0 = \frac{\pi\ell\sqrt{\mu_{AdS}}}{2G_3} = C_v, \\ T_H &= \frac{\sqrt{\mu_{AdS}}}{2\pi\ell} = \left[\frac{G_3}{\pi^2\ell^2}\right]S_0, \quad E = F + T_H S_0 = \frac{\mu_{AdS}}{8G_3} = \mathcal{M}_{AdS} = M_{AdS}, \end{aligned} \quad (5)$$

where G_3 is the 3D Newton constant. Here we obtain a positive specific heat. In this case we have the ADM mass $M_{AdS} = \mathcal{M}_{AdS}$ which is consistent with that from the Brown-York approach in Ref. [19].

III. 3D SCHWARZSCHILD-DE SITTER SPACETIME

The 3D Schwarzschild-de Sitter spacetime is given by [20]

$$ds_{SdS}^2 = -h_{SdS}(r)dt^2 + \frac{1}{h_{SdS}(r)}dr^2 + r^2d\theta^2 \quad (6)$$

where $h_{SdS}(r)$ is given by

$$h_{SdS}(r) = 1 - \mu_{SdS} - \frac{r^2}{\ell^2} \quad (7)$$

with a reduced mass $\mu_{SdS} = 8G_3\mathcal{M}_{SdS}$. The cosmological horizon is located at

$$r_{CH} = \ell \sqrt{1 - \mu_{SdS}}. \quad (8)$$

In this case we have to put a particle with a small mass at $r = 0$ ($r_{EH} < \ell$, $0 < \mu_{SdS} < 1$). This describes a conical defect spacetime with a deficit angle $2\pi(1 - \sqrt{1 - \mu_{SdS}})$, indicating a world with a positive cosmological constant and a pointlike object with mass \mathcal{M}_{SdS} [21]. In the case of $\mu_{SdS} > 1$, any cosmological horizon does not exist.

Interesting thermodynamic quantities for the cosmological horizon are given by [22]

$$F = -\left[\frac{1 - \mu_{SdS}}{8G_3}\right], \quad S_0 = \frac{\pi\ell\sqrt{1 - \mu_{SdS}}}{2G_3} = C_v, \quad (9)$$

$$T_H = \frac{\sqrt{1 - \mu_{SdS}}}{2\pi\ell} = \left[\frac{G_3}{\pi^2\ell^2}\right]S_0, \quad E = F + T_H S_0 = \frac{1 - \mu_{SdS}}{8G_3} = M_{SdS}.$$

We obtain a positive specific heat only for a small massive particle ($\mu < 1$) inside the cosmological horizon. In other words, this gravitating system composed of the cosmological horizon $r_{CH} < \ell$ and a massive particle with mass \mathcal{M}_{SdS} is thermodynamically stable because $F < 0$ when $\mu < 1$. If $\mathcal{M}_{SdS} = 0$, we recover an ADM mass of $M_{dS} = 1/8G_3$ for the pure de Sitter space with a cosmological horizon at $r_{CH} = \ell$. In the case of $\mathcal{M}_{SdS} \neq 0$, the ADM mass ($M_{SdS} = 1/8G_3 - \mathcal{M}_{SdS}$) of this system is less than that of the pure de Sitter space. Here we address why a particle with $\mathcal{M}_{SdS} > 0$ inside the cosmological constant contributes to M_{SdS} as a negative one. The reason is that even if the matter of a pointlike object has a positive energy, the binding energy to the gravitational de Sitter background can make it negative. That is, a conical defect at $r = 0$ swallows up a part of the spacetime and thus appears to reduce the net amount of the positive energy stored in the cosmological constant. In this sense we no longer take \mathcal{M}_{SdS} as a true mass (energy) in SdS space. The true mass is given by the ADM mass M_{SdS} . As a result, the presence of a pointlike object inside the cosmological horizon decreases the size of the cosmological horizon in compared with the pure de Sitter case.

IV. 3D KERR-DE SITTER SPACE

When introducing a negative mass in higher-dimensional de Sitter space one could find a non-spherical horizon. We call this the topological de Sitter space [23]. However, in three dimensions we could not construct various horizon geometries because one-dimensional horizon is locally flat and is equivalent to each other. Here we introduce the Kerr-de Sitter space with $J = 0$ [24]

$$ds_{KdS}^2 = -h_{KdS}(r)dt^2 + \frac{1}{h_{KdS}(r)}dr^2 + r^2d\theta^2, \quad (10)$$

where $h_{KdS}(r)$ is given by

$$h_{KdS}(r) = -\tilde{\mu}_{KdS} - \frac{r^2}{\ell^2} = \mu_{KdS} - \frac{r^2}{\ell^2} \quad (11)$$

with a negative reduced mass $\tilde{\mu}_{KdS} \equiv -\mu_{KdS} = -8G_3\mathcal{M}_{KdS}$ in compared with the SdS case in Eq. (7). The cosmological horizon is given by

TABLE I. Summary of specific heat and entropy for BTZ black hole, KdS and SdS spaces. Ground-state horizon information ($\mu = 0$) is added.

3D thermal system	$C_v (= S_0)$	horizon if $\mu = 0 (\mu \neq 0)$
BTZ with $J = 0$	+	N
KdS with $J = 0$	+	N
SdS	+ if $\mu < 1$	Y(\leftarrow)

$$r_{CH} = \ell \sqrt{\mu_{KdS}}. \quad (12)$$

Thermodynamic quantities are calculated as

$$F = -\frac{\mu_{KdS}}{8G_3}, \quad S_0 = \frac{\pi \ell \sqrt{\mu_{KdS}}}{2G_3} = C_v, \quad (13)$$

$$T_H = \frac{\sqrt{\mu_{KdS}}}{2\pi \ell} = \left[\frac{G_3}{\pi^2 \ell^2} \right] S_0, \quad E = F + T_H S_0 = \frac{\mu_{KdS}}{8G_3} = \mathcal{M}_{KdS} = M_{KdS}.$$

Here we obtain a positive specific heat, which means that the KdS space is thermodynamically stable. We observe that there is no difference between Eqs. (5) and (13).

Let us ask of how to understand the positivity of ADM mass ($M_{KdS} > 0$) in the presence of a pointlike object with a negative energy. The topological de Sitter space in five-dimensions has a positive mass and a timelike singularity which remains within a single cosmological horizon. Hence it satisfies the mass bound conjecture: any asymptotically de Sitter space whose mass exceeds that of pure de Sitter space contains a cosmological singularity [21]. In the case of 3D KdS space, there is no cosmological singularity. However, the negative energy of a pointlike object at $r = 0$ gives the ADM mass of $M_{KdS} = \mathcal{M}_{KdS}$ in asymptotically de Sitter space. As a result, one finds that $r_{CH}^{KdS} = \ell \sqrt{\mu_{KdS}} > r_{CH}^{dS} = \ell$ if $\mu_{KdS} > 1$. This corresponds to a conical excess. If $\mu_{KdS} < 1$, one finds a conical defect.

V. CORRECTION TO BEKENSTEIN-HAWKING ENTROPY

We summarize our result in TABLE I. In this section we make corrections to the Bekenstein-Hawking entropy according to the formula of Eq.(1). For all cases, one finds $C_v = S_0$, $T_H = \left[\frac{G_3}{\pi^2 \ell^2} \right] S_0$ without any approximation. Here we require a small SdS solution for obtaining a positive specific heat. All logarithmic corrections to the Bekenstein-Hawking entropy are given by a single relation as

$$S = S_0 - \frac{3}{2} \ln S_0 + \dots \quad (14)$$

This means that any thermodynamically stable system gets a logarithmic correction to the Bekenstein-Hawking entropy due to thermal fluctuation around the equilibrium state. Here we confirm that this correction is universal for all 3D gravitational systems with $J = 0$. Further we wish to mention that all thermal quantities of the KdS spaces take the same form as those from the BTZ black holes. This implies that there is no distinction between

the event horizon of BTZ black hole and the cosmological horizon of Kerr-de Sitter space. That is, a pointlike object with a positive energy inside the event horizon is equivalent to a pointlike object with a negative energy inside the cosmological horizon. We note that even for $\mu_{SdS} = 0$, the SdS space has a cosmological horizon. If $\mu_{SdS} \neq 0$, the size of the cosmological horizon is decreased.

VI. ADS/CFT AND DS/ECFT CORRESPONDENCE

The holographic principle means that the number of degrees of freedom associated with the bulk gravitational dynamics is determined by its boundary spacetime without gravity. Let us first discuss the AdS black hole. The AdS/CFT correspondence represents a realization of this principle [25]. Brown and Henneaux have shown that asymptotic symmetries in 3D AdS spacetime are described by a pair of Virasoro algebras with central charges $c = \bar{c} = 3\ell/2G_3$ [26]. The generators of Brown-Henneaux Virasoro algebras are simply Hamiltonian and momentum constraints of 3D gravity smeared against appropriate vector fields. For BTZ black hole with $J = 0$, these are $L_0 = M\ell/2 = \bar{L}_0$ with the ADM mass $M = \mathcal{M}_{AdS}$ [3]. In order to calculate the entropy on the boundary spacetime, we use the Cardy formula for a pair to yield

$$S_{CFT} = 2\pi\sqrt{\frac{cL_0}{6}} + 2\pi\sqrt{\frac{\bar{c}\bar{L}_0}{6}} = \frac{\pi r_{EH}}{2G_3} = S_0 \quad (15)$$

which establishes the AdS/CFT correspondence for the entropy. According to the Carlip's work [3], the density of states can be approximated as

$$\rho(L_0, \bar{L}_0) = \frac{8G_3\ell}{r_{EH}^3} \exp\left[\frac{\pi r_{EH}}{2G_3}\right]. \quad (16)$$

Thus the entropy correction due to thermal fluctuation of the CFT system leads to

$$S = S_0 - 3 \ln S_0 + \dots \quad (17)$$

Comparing the above with the bulk correction in Eq. (14), one finds logarithmic term that differs from it by a factor of 2. However, this difference does not seem to be serious because other approaches suggest a correct factor of 3/2 [7,3].

For SdS and KdS spaces, according to Strominger [27], the dS/ECFT proposal is useful for calculating the logarithmic corrections to the boundary entropy. We note that 3D de Sitter space can be represented by the group manifold of $SL(2, \mathbb{C})/SL(2, \mathbb{R})$. The asymptotic symmetry group of de Sitter space, subject to the boundary conditions, is the two-dimensional Euclidean conformal group $SO(3,1)$, which contains the isometry group $SL(2, \mathbb{C})$ as a subgroup. Indeed, the Brown-Henneaux analysis can be formally continued to arrive at Virasoro algebras with central charges $c = \bar{c} = 3\ell/2G_3$ [21]. Its dual theory might be a Lorentzian CFT with Euclidean signature Virasoro algebras. Actually the ADM masses M_{SdS}, M_{KdS} in Eqs.(9) and (13) are consistent with those computed with the Euclidean signature on the spacelike boundary. Thus we have generators as $L_0 = M\ell/2 = \bar{L}_0$ where the ADM mass is given by $M_{SdS} = 1/8G_3 - \mathcal{M}_{SdS}$ for the SdS case and $M_{KdS} = \mathcal{M}_{KdS}$ for

the KdS space. At this stage, we remind the reader that for an observer near infinity, there is no actual difference between the event horizon of AdS black hole and the cosmological horizon of SdS and KdS space. The difference results in the precise value of a conserved quantity, the ADM mass. Hence we expect that their dual ECFTs exist on the boundary. The Cardy formula provides

$$S_{ECFT} = 2\pi\sqrt{\frac{cL_0}{6}} + 2\pi\sqrt{\frac{\bar{c}\bar{L}_0}{6}} = \frac{\pi r_{CH}}{2G_3} = S_0 \quad (18)$$

which establishes the dS/ECFT correspondence for the entropy. Following the dS/ECFT correspondence, their logarithmic corrections will be given by the same form as in Eq. (17).

On the other hand, a complete Cardy formula for the asymptotic density of states of a unitary (E)CFT is given by

$$S_{CCFT} = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)} + 2\pi\sqrt{\frac{\bar{c}}{6}\left(\bar{L}_0 - \frac{\bar{c}}{24}\right)}. \quad (19)$$

A naive application of this formula with the same central charge and generators fails to find the Bekenstein-Hawking entropy S_0 , in contrast with the use of the Cardy-formula in Eq. (15). However, in the case of $M_{AdS/KdS}\ell/2 \gg c/24 = \ell/16G_3$, the complete Cardy formula is given approximately

$$S_{CCFT} \simeq \frac{\pi\ell\sqrt{\mu_{AdS/KdS}}}{2G_3} = S_0. \quad (20)$$

Hence we find that the Cardy formulae in Eqs. (15) and (18) are considered as approximate forms to the complete Cardy formula in Eq. (19).

VII. DISCUSSION

First of all, we summarize our main result. We find that all thermal quantities of KdS space take the same form as those from BTZ black hole. This means that there is no difference between the event horizon and cosmological horizon in 3D gravity systems with the single horizon.

We have a few of comments in order. Any ADM mass for a gravitational system with a single horizon including the BTZ black hole with $J = 0$, the Kerr-de Sitter space with $J = 0$, and SdS solution is always positive. In three dimensions, any system with a single horizon has a positive specific heat and its canonical ensemble is thermodynamically stable. Furthermore, this system has an equality of $C_v = S_0$, which dictates that it has a single horizon in three dimensions. Thus we find the same form for logarithmic corrections to the Bekenstein-Hawking entropy and the Cardy formula.

Concerning A(dS)/(E)CFT correspondences, we note that these are applicable for the event horizon for AdS black hole and the cosmological horizons for KdS and SdS spaces. Boundary systems have the same central charge of $c = \bar{c} = 3\ell/2G$ with generators $L_0 = \bar{L}_0 = M\ell/2$ where M is one ADM mass for AdS black holes, KdS and SdS spaces. In general the boundary entropy is equal to the Bekenstein-Hawking entropy and its logarithmic correction is given by Eq. (17).

Finally we wish to remark the difference between 3D and 5D gravity systems. In 5D gravity, the parameter k is introduced to classify the 3D horizon geometry: $k = 0, 1, -1$ denote the flat, elliptic, and hyperbolic spaces with constant curvature. That is, it is necessary to classify topological AdS black holes and topological de Sitter spaces. In 3D gravity, however, the horizon is one-dimensional space and it is locally flat. Hence the parameter like k is irrelevant to 3D gravity systems. Even if k is introduced, it could be absorbed into μ by the redefinition. Therefore, in this work we investigated thermal properties of three known examples: BTZ black hole with $J = 0$, Kerr-de Sitter space with $J = 0$, and Schwarzschild-de Sitter space. In other words, the BTZ black hole is representative of 3D topological AdS black holes, while the Kerr-de Sitter is representative of 3D topological Sitter spaces.

ACKNOWLEDGMENTS

The author thanks Rong-Gen Cai and Mu-In Park for helpful discussions. This work was supported in part by KOSEF, Project No. R02-2002-000-00028-0.

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